ON THE STABILIZATION OF THE STEADY STATE MOTION OF A NONLINEAR CONTROL SYSTEM IN THE CRITICAL CASE OF A DOUBLE ZERO ROOT

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The general problem of the stabilization to asymptotic stability [1] of the steady state motions of nonlinear control systems is considered in paper [2]. The present paper considers the conditions of stabilization of a nonlinear system in the critical case of a double zero root.

1. Let us consider the control system

$$dy / dt = f(t, y, w) \qquad (y \in \{R^{n+2}\},\ w \in \{R^m\}) \tag{1.1}$$

where f is a given vector function; y is the $(n+2)$-dimensional vector of the phase coordinates of the system; w is the m-dimensional vector of the control. The vector y is subject to small perturbations x, such that in (1.1)

$$y = y^*(t) + x(t) \tag{1.2}$$

where $y^*(t)$ is a given motion caused by the control $w^*(t, y^*(t))$. We shall denote

$$u = w - w^* \tag{1.3}$$

Substituting (1.2) and (1.3) into (1.1) and expanding the right-hand sides in powers of x, u, we get the following Eqs. of the perturbed motion

$$\frac{dx}{dt} = \sum_{i=1}^{n+2} \left(\frac{\partial f}{\partial y_i} + \sum_{j=1}^{m} \frac{\partial f}{\partial w_j} \frac{\partial w_j^*}{\partial y_i} \right) x_i + \sum_{j=1}^{m} \frac{\partial f}{\partial w_j} u_j + g(t, x, u) \tag{1.4}$$

Here the derivatives are computed along the motion $y = y^*(t)$, $w = w^*(t)$; $g(t, x, u,)$ represents the terms of order higher than the first in x, u. We shall assume that the order of smallness of u is not lower than that of x, i.e. $u(t, 0) = 0$ $(t \geqslant 0)$:

$$|u_j(t, x'') - u_j(t, x')| \leqslant \alpha \sum_{i=1}^{n+2} |x_i'' - x_i'| \quad \begin{pmatrix} j = 1, \ldots, m \\ \alpha > 0, \text{ const} \end{pmatrix}$$

for small x', x''.

If for $u = 0$ the unperturbed solution $x = 0$ of the system (1.4) is unstable, then there appears the problem of the stabilization of the motion (1.1), i.e. the problem of the choice of a control $u(t, x)$ such that if it is substituted into (1.4) the zero solution $x = 0$ by asymptotically stable according to Liapunov [1].

We shall assume that the unperturbed solution has reached the steady state. Thus we shall consider the system

$$dx / dt = Ax + Bu + g(x, u) \tag{1.5}$$

where x is the $(n+2)$-dimensional vector of the perturbation, u is the m dimensional vector of the control, which we shall consider as unaffected by disturbances; A and B are constant matrices of appropriate dimensions. We shall assume that all the coefficients of Eq. (1.5) are real and that $g(x, u)$ is a function analytic in x and u of an order which cannot be lower than the second.

2. Let it be the critical case of a double zero root [2].

Let us consider the first case in which the zero root does not make zero at least one of the $(n + 1)$ order minors of the characteristic determinant of the system of equations of perturbed motion under consideration or, what is the same, one group of solutions of the system first order approximation corresponds to the root.

Then, by means of a nondegenerate coordinates change, the matrix of which can be constructed, following [1 and 2] by taking ξ, η for new coordinates and assuming $x_i = v_i$ $(i = 1, 2,\ldots,n)$, we may bring the system (1.5) into the form

$$\frac{d\xi}{dt} = \eta + X(\xi, \eta, v, u), \qquad \frac{d\eta}{dt} = Y(\xi, \eta, v, u), \qquad \frac{dv}{dt} = A_0 v + B_0 u + Z(\xi, \eta, v, u) \qquad (2.1)$$

Here ξ, η are the scalars; v is the n-dimensional vector of components v_s; A_0 and B_0 are constant matrices of order $n \times n$ and $n \times m$; Z is a vector-function of components Z_s; X, Y, Z_s are analytical nonlinearities in ξ, η, v, u.

The stabilization problem for the system of Eqs. (1.5) with respect to the variables x_i $(i = 1, 2,\ldots,n + 2)$ is equivalent to the same problem for the system (2.1) with respect to the variables ξ, η, v_i $(i = 1, 2,\ldots,n)$. Let us consider the system

$$dv / dt = A_0 v + B_0 u$$

which satisfies the stabilizability condition, indicated in the theorem 3.1 of paper [2]. (See also [3 to 5]). Consequently, on the basis of the Theorem 3.1 there exists a linear control

$$u_0(v) = Pv \qquad (2.2)$$

such that the trivial solution of the system of linear equations with constant coefficients

$$dv / dt = (A_0 + B_0 P)\, v$$

is asymptotically stable. Here, P is a certain $m \times n$ constant matrix.

Let us search for the system (2.1) a nonanalytic control of the form

$$u^j(\xi, |\xi|, v) = u_0^j(v) + \sum_{s+k=1}^{\infty} \alpha_{sk}^j \xi^s |\xi|^k \qquad \left(\begin{matrix} j = 1, 2, \ldots, m \\ s \geqslant 0,\ k \geqslant 0 \end{matrix}\right) \qquad (2.3)$$

For $k = 0$ an analytic control is obtained. The use of the nonanalytic control increases notably the stabilization possibilities.

In agreement with Liapunov's method let us consider the system

$$A_0 v + B_0 u + Z(\xi, \eta, v, u) = 0 \qquad (2.4)$$

If the control (2.3) is substituted into (2.4), its functional determinant with respect to v_i for zero values of ξ, η, v_i is different from zero [2] (See also [3 and 4]). Therefore there exists a solution of the system (2.4) in the neighborhood of the origin of the coordinates ($(\xi = 0, \eta = 0, v = 0)$, which can be represented by the series

$$v_i^0 = \sum_{s+k+l=2}^{\infty} a_{skl}^i \xi^s |\xi|^k \eta^l \qquad \left(\begin{matrix} i = 1, 2, \ldots, n \\ s \geqslant 0,\ k \geqslant 0,\ l \geqslant 0 \end{matrix}\right) \qquad (2.5)$$

where the coefficients a_{skl}^i are functions of α_{sk}^j.

After substitution of (2.2) and (2.3), into (2.1) the functions $Y(\xi, |\xi|, \eta, v)$, $Z(\xi, |\xi|, \eta, v)$ of the right-hand side of the second and third equations of that system have the form

$$Y(\xi, |\xi|, \eta, 0) = Y^0(\xi, |\xi|) + \eta(Y^1)(\xi, |\xi|) + \ldots \qquad (2.6)$$

$$Z(\xi, |\xi|, \eta, 0) = Z^{(0)}(\xi, |\xi|) + \eta Z^{(1)}(\xi, |\xi|) + \ldots \qquad (2.7)$$

Here the terms omitted do not include η at a power lower than the second. The coefficients of the expansion of the functions $Y^{(i)}$, $Z^{(i)}$ are expressed in a specific manner by the coefficients α_{sk}^j .

We shall denote the powers of the smallest terms in the expansion of the functions $Y^{(0)}$ $Y^{(1)}$, $Z^{(0)}$, $Z^{(1)}$ respectively by p, q, p_s, q_s $(p \geqslant 2,\ q \geqslant 1,\ p_s \geqslant 2,\ q_s \geqslant 1;\ s = 1, 2,\ldots,n)$.

The series (2.6) and (2.7) appear as functions Y, Z in the right-hand side of the system (2.1) after Liapunov's transformation

$$v_i = v_i^\circ + v_i^* \qquad (i = 1, 2, \ldots, n) \qquad (2.8)$$

if we take $v_i^* \equiv 0$.

For such a choice of the control (2.3) and after substitution of (2.8) into the system (2.1) the right-hand sides of the transformed system can have discontinuities on the surface $\xi = 0$. Thus, the convergence principle [8] is also valid in that case.

By means of the transformations (2.8) which do not change the form of the system (2.1) we may always obtain a new system. [6 and 7] which depends on the structure of the right-hand sides and can be referred to one of the following cases

1) If $Y^{(0)}(\xi, |\xi|) \equiv 0$, $Y^{(1)}(\xi, |\xi|) \not\equiv 0$, then $Z^{(0)}(\xi|\xi|) \equiv 0, q_s > q$ $(s = 1, \ldots n)$

2) If $Y^{(0)}(\xi, |\xi|) \equiv Y^{(1)}(\xi|\xi|) \equiv 0$, then $Z^{(0)}(\xi|\xi|) \equiv Z^{(1)}(\xi|\xi|) \equiv 0$

3) If $Y^{(0)}(\xi, |\xi|) \not\equiv 0$, then $p_s > p$; thus $q_s > q$, if $p > q$ or $q_s \geqslant p$, if $q \geqslant p$ $(s = 1, 2, \ldots, n)$

Furthermore, it can always be assumed [6 and 7] that $X(\xi, 0, 0) \equiv 0$. We shall examine each of these cases in succession.

Next, we shall follow the known stability theory in the critical case of a double zero root [6 and 7] and also the developed stabilization theory [2]. The validity of the statements made below follows from Liapunov's theorems and Chetaev's instability theorem [9], for a quasi-analytic system.

3. Let us assume the conditions (1) are satisfied. Let us denote the sum of the coefficients of all the even functions of the ensemble of the terms $Y^{(1)}(\xi, |\xi|)$ in (2.6) by b_k and analogously the sum for the odd functions by b_k^* $(k \geqslant 1)$, which depend in a definite manner on the coefficients a_{sk}^j. Here and from there on, the indices of the coefficients of the function $Y^{(1)}(\xi, |\xi|)$ will also denote the powers of the corresponding terms.

If the coefficients a_{sk}^j of (2.3) can be chosen such that the condition

$$b_q < -|b_q^*| \qquad (q = \min k \geqslant 1) \tag{3.1}$$

is satisfied, then the stabilization of the system (2.1), and consequently (1.5) are guaranteed by the control (2.3). The unperturbed motion of the system (2.1) is stable, and each perturbed motion, sufficiently close to the unperturbed one, asymptotically comes closer to some steady state motion $\xi = a$, $\eta = v_j = 0$.

If the condition (3.1) is not fulfilled, but the weaker condition $b_q = -|b_q^*|$ is satisfied, then one should consider the ensemble $Y^{(1)}(\xi, |\xi|)$ of the terms of the next order. Thus if the remaining coefficients a_{sk}^j can be chosen such that after a step $k > q$ the condition

$$b_k < -|b_k^*| \qquad \text{or} \qquad b_k^* b_q^* < 0, \quad b_k < |b_k^*|$$

is satisfied, if the relations

$$b_{q+i}^* b_q^* > 0, \; b_{q+i} = -|b_{q+i}^*| \qquad \text{or} \qquad b_{q+i}^* b_q^* < 0, \; b_{q+i} = |b_{q+i}^*|$$
$$(i \leqslant k - q - 1; \quad i = 0, 1, \ldots, K) \tag{3.2}$$

are satisfied, then the stabilization of the system (2.1) is assured and the control (2.3) is constructed.

If for any possible values of a_{sk}^j the coefficients of the smaller terms satisfy the condition $b_q > 0$ or $|b_q^*| > -b_q$ for $b_q \leqslant 0$, or if for some $k > q$ and the conditions (3.2) one of the conditions

$$b_q^* b_k^* < 0, \quad b_k > |b_k^*|; \qquad b_k^* b_q^* > 0, \quad b_k > -|b_k^*|$$

is satisfied, then the stabilization is not possible by means of the control (2.3).

In the case (2) the system (2.1) cannot be stabilized by the given method.

Let us consider the case (3). Let us point out here, that under certain conditions we shall assume further on that $X \equiv 0$, and that the functions $Z_s(\xi, |\xi|, \eta, 0)$ (2.7) do not contain terms of any arbitrary high order. This is always possible by means of some transformation [6 and 7] which does not change the form of the system (2.1). Let us denote the sum of the coefficients for all even functions of the ensemble of the terms $Y^{(0)}(\xi, |\xi|)$ by a_s and the analogous sum of the odd terms by a_s^* $(s \geqslant 2)$; these coefficients are functions of the a_{sk}^j.

If, by an appropriate choice of the a_{sk}^j it is possible to obtain that for

$$k < s, \quad k + s = 2\nu + 1 \tag{3.3}$$

the conditions

$$a_p^* < -|a_p|, \quad b_q < -|b_q^*| \quad (p = \min s \geqslant 2, \; q = \min k \geqslant 1) \tag{3.4}$$

be fulfilled, then the control (2.3) assures the stabilization of the system (2.1).

If the conditions (3.4) are not fulfilled, but the conditions

$$a_p^* = -|a_p|, \qquad b_q = b_q^* < 0 \tag{3.5}$$

are at least satisfied, then one should consider the ensemble of the terms of the next order Thus the control (2.3) is obtained, if after some step for $s > p$, $k > q$ and (3.3) the conditions

$$a_s^* < -|a_s|, \qquad a_s a_p < 0, \quad a_s^* < |a_s|, \qquad b_k < b_k^*$$

are satisfied, and furthermore

$$b_{q+i} = b_{q+i}^* \quad (\text{for } i = 0)$$

$$a_{p+j} a_p > 0, \ a_{p+j}^* = -|a_{p+j}| \quad \text{or} \quad a_{p+j} a_p < 0, \ a_{p+j}^* = |a_{p+j}| \tag{3.6}$$

for each

$$i \leqslant \kappa - q - 1, \quad j \leqslant s - \nu - 1 \qquad (i = 0, 1, \ldots, K_1;\ j = 0, 1, \ldots, K_2)$$

Note 3.1. If only one of the conditions (3.4) is not satisfied, but the corresponding condition from (3.5) is met, then one must search for approximations for the corresponding coefficients, (3.3) being satisfied.

Note 3.2. If the stabilization of the system (2.1) is assured because the conditions (3.4) are met, it is sufficient to take a control (2.3) of the form

$$u^j = u_0^j(v) + \alpha_{10}^j \xi + \alpha_{01}^j |\xi|$$

The stabilization of the system (2.1) by the control (2.3) is not possible if for any choice of the coefficients α_{sk}^j one gets

$$a_p^* > 0; \quad |a_p| > -a_p^* \quad \text{for} \quad a_p^* < 0$$

$$a_p^* < -|a_p|, \quad b_q > |b_q^*| \quad \text{for (3.3)} \tag{3.7}$$

or, if after some step $s > p$, $k > q$ and (3.3) at least one of the following inequalities is satisfied

$$a_s a_p < 0, \quad a_s^* > |a_s|; \quad a_s a_p > 0, \quad a_s^* > -|a_s|$$

$$b_k > b_k^* \quad \text{for (3.6)}, \quad b_{q+i} = b_{q+i}^* \quad \text{for } i \leqslant k - q - 1 \quad (i = 0, 1, \ldots, K_1)$$

$$b_q = b_q^* > 0 \quad \text{for } i = 0.$$

Let us note that if only one of the conditions (3.7) is not satisfied it is necessary to search for approximations of the corresponding coefficients. The systems (2.1) are not stabilized by this method either if

$$a_p^* < -|a_p|, \quad b_q^* > |b_q| \quad \text{for } q < 1/2\,(p-1) \quad \text{or}$$

$$(b_q + b_q^*)^2 + 4(q+1)(a_p + a_p^*) \geqslant 0 \quad \text{for } q = 1/2\,(p-1)$$

Now let us consider the case when

$$a_p^* < -|a_p|, \qquad q > 1/2\,(p-1)$$

$$(b_q + b_q^*)^2 + 4(q+1)(a_p + a_p^*) < 0, \qquad q = 1/2\,(p-1)$$

Let $p = 2m - 1$ ($m \geqslant 1$ is an integer). Let us bring to consideration the functions of Liapunov Cs and Sn, determined by the relations [1, 6 and 7]

$$\mathrm{Cs}^{2m}\theta + m\,\mathrm{Sn}^2\theta = 1, \qquad \mathrm{Cs}\,0 = 1, \qquad \mathrm{Sn}\,0 = 0$$

$$d\,\mathrm{Cs}\,\theta / d\theta = -\mathrm{Sn}\,\theta, \qquad d\,\mathrm{Sn}\,\theta / d\theta = \mathrm{Cs}^{2m-1}\theta$$

Let us assume the construction such that the transformation (2.8) is continuous. We transform the system (2.1) by means of the substitution

$$\xi = r\,\mathrm{Cs}\,\theta, \qquad \eta = -r^m \mathrm{Sn}\theta$$

Then eliminating time t from the first two equations, we get

$$dr/d\theta = r^2 R_2(\theta) + r^3 R_3(\theta) + \ldots \quad \text{for } q > m - 1 \tag{3.8}$$

Here $R_k(\theta)$ ($k = 2, 3, \ldots$) are functions with respect to Cs θ, Snθ of the form Y^i in (2.6) with coefficients depending on α_{sk}^j. The solution of Eq. (3.8) is sought in the form

$$r = c + c^2 u_2(\theta) + c^3 u_3(\theta) + \ldots \tag{3.9}$$

with the initial condition $r(0, c) = c$. Substituting (3.9) into (3.8) we get for $u_k(\theta)$

$$du_2 / d\theta = R_2(\theta) = F_2(\theta), \qquad du_3 / d\theta = R_3(\theta) + 2R_2(\theta)\,u_2(\theta) = F_3(\theta), \ldots$$

Let us assume that some of the coefficients u_2, u_3 are nonperiodic. Let $u_m(\theta)$ be the first nonperiodic one

$$u_m = g\theta + G(\theta)\left(G(\theta) = G(\theta + 2\pi), \quad g = \frac{1}{2\pi}\int_0^{2\pi} F_m(\theta)\,d\theta \neq 0\right)$$

Then the control (2.3) assures the stabilization of the system, if the coefficients α_{sk}^j

can be chosen such that the condition $g < 0$ be satisfied.

If $g > 0$, it is not possible to stabilize the system using this method. If the conditions

$$q = m - 1, \quad (b_q + b_q^*)^2 + 4m(a_p + a_p^*) < 0$$

are satisfied, the equation analogous to Eq. (3.8) has the form

$$dr / d\theta = r R_1(\theta) + r^2 R_2(\theta) + r^3 R_3(\theta) + \ldots$$

where $R_k(\theta)$ $(k = 1, 2, \ldots,)$ are functions having the same structure as Eq. (3.8).

In that case the solution of the equation is sought in the form

$$r = cu_1(\theta) + c^2 u_2(\theta) + c^3 u_3(\theta) + \ldots$$

and the stabilization conditions are analogous to the previous ones.

If we take $k = 0$ in (2.3) we get an analytic control of the form

$$u^j = u_0^j(v) + \sum_{s=1}^{\infty} \alpha_{s0}^j \xi^s$$

In that case, the stabilization of the system is possible, and the solution of the problem can be obtained by adapting known investigation methods of the stability theory [6 and 7].

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